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ON THE HEAT CONTENT OF A POLYGON

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Abstract. Let D be a bounded, connected, open set in Euclidean space $\mathbb{R}^2$ with polygonal boundary. Suppose D has initial temperature 1 and the complement of D has initial temperature 0. We obtain the asymptotic behaviour of the heat content of D as time $t \downarrow 0$. We then apply this result to compute the heat content of a particular fractal polyhedron as $t \downarrow 0$.

1. Introduction

The conduction of heat or the diffusion of matter through a solid body is of importance in the physical and engineering sciences. The classic reference of Carslaw and Jaeger, [9], analyses many examples and applications. The mathematical tools used in [9] are centred around separation of variables and Laplace transforms and, in many cases, require properties of special functions. From a mathematical point of view, the heat equation, heat content and heat trace link the underlying geometry of the manifold and its boundary and boundary conditions to the spectral resolution of the Laplace operator. Over the last few decades, a considerable amount of progress has been made in understanding the asymptotic behaviour of the heat content for small time t, see [12].

It was discovered by Preunkert, [17], that even in the absence of boundary conditions the heat content of a ball B in Euclidean space $\mathbb{R}^m$ which is at initial temperature 1, while $\mathbb{R}^m - B$ has initial temperature 0, has non-trivial asymptotic behaviour as $t \downarrow 0$. For small t, the initial condition on the complement of B acts in a similar way to a Dirichlet 0 boundary condition. This was subsequently stated for bounded, open sets with $C^{1,1}$ boundary in [15], and proved in [16]. The discussion is simplified by the fact that the heat kernel on $\mathbb{R}^m$ is known explicitly. The general situation for the heat content of a compact subdomain Ω in a compact Riemannian manifold M was examined in [4]. The tools of pseudo-differential calculus used there rely heavily upon the smoothness assumptions on the boundary. Two-sided estimates for the heat content of non-compact sets in $\mathbb{R}^m$ were obtained in [3]. These estimates are very different from the ones where Dirichlet 0 boundary conditions are imposed. See [1] and [2].

In this paper we denote the fundamental solution of the heat equation on $\mathbb{R}^m$ by

$$p(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},$$

and for an open set $D \subset \mathbb{R}^m$, we define

$$u_D(x; t) = \int_D dy\, p(x, y; t). \tag{1}$$

Then $u_D(x; t)$ satisfies the heat equation on $\mathbb{R}^m$

$$\Delta u_D = \frac{\partial u_D}{\partial t}, \quad x \in \mathbb{R}^m,\, t > 0, \tag{2}$$

(see Chapter 2 in [10]) and

$$\lim_{t\downarrow 0} u_D(x; t) = \mathbb{1}_D(x), \quad x \in \mathbb{R}^m - \partial D, \tag{3}$$

where ∂D is the boundary of D.

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We define the *heat content of D in $\mathbb{R}^m$ at t* by

$$H_D(t)=\int_D dx\,u_D(x;t).$$

So by (1),

$$H_D(t)=\int_D dx\int_D dy\,p(x,y;t). \tag{4}$$

We denote the Lebesgue measure of a measurable set $A\subset\mathbb{R}^m$ by $|A|$, its perimeter by $\mathcal{P}(A)$, and its $(m-1)$-dimensional Hausdorff measure by $\mathcal{H}^{m-1}(A)$.

If D is a bounded, open set in $\mathbb{R}^m$, $m\ge 2$, with $C^{1,1}$ boundary ∂D, then Theorem 2.4 of [16] implies that

$$H_D(t)=|D|-\mathcal{P}(D)\frac{t^{1/2}}{\sqrt{\pi}}+o(t^{1/2}),\ t\downarrow 0. \tag{5}$$

In [5], we obtained explicit bounds for $H_D(t)$ for bounded, open sets D in Euclidean space with $C^{1,1}$ boundary which are uniform in t and in the geometric data of D. These bounds imply that

$$H_D(t)=|D|-\mathcal{P}(D)\frac{t^{1/2}}{\sqrt{\pi}}+O(t),\ t\downarrow 0. \tag{6}$$

We observe that if K is a closed set in $\mathbb{R}^m$ with $|K|=0$ then by the definition of the perimeter (see [11]) and (4),

$$|D-K|=|D|,\quad \mathcal{P}(D-K)=\mathcal{P}(D),\quad H_{D-K}(t)=H_D(t). \tag{7}$$

The observations in (7) suggest that (5) holds for all open sets D with finite perimeter and finite Lebesgue measure. The proof of such a statement is well beyond the scope of this paper.

In this paper, we focus on the heat content of a bounded, connected, open set $D\subset\mathbb{R}^2$ with polygonal boundary. We introduce some notation and then present the main result: Theorem 1. Let $\gamma_1,\cdots,\gamma_n$ denote the interior angles of ∂D. Each such angle γ_j is supported by two edges provided $\gamma_j<2\pi$. By (7), we may exclude angles 2π. We label the corresponding vertices by $V_1,\cdots,V_n$ and note that these n vertices need not be pairwise disjoint. Let W_j denote the infinite wedge of angle γ_j with vertex V_j such that $W_j\cap D\neq\emptyset$ and the boundary of the wedge contains the two edges which are adjacent to V_j and have an angle γ_j. Let

$$\gamma=\{\gamma_i\,:\,(\sin\gamma_i)^2\le(\sin\gamma_j)^2\ \text{ for all } j\in\{1,2,\cdots,n\}\}. \tag{8}$$

For $r>0$, we also define the open sector

$$B_j(r)=\{x\in W_j\,:\,d(x,V_j)<r\} \tag{9}$$

and

$$R=\frac{1}{2}\sup\left\{r\,:\,B_\ell(r)\cap B_j(r)=\emptyset \text{ for all } \ell\neq j,\ \bigcup_{k=1}^n B_k(r)\subset D\right\}. \tag{10}$$

Theorem 1. *Let $D\subset\mathbb{R}^2$ be a bounded, connected, open set with polygonal boundary ∂D with γ and R as defined in* (8) *to* (10). *Then as $t\downarrow 0$,*

$$\begin{aligned}H_D(t)=|D|-\mathcal{P}(D)\frac{t^{1/2}}{\sqrt{\pi}}&+\sum_{j=1}^n g(\gamma_j)t\\&+\sum_{j,\ell\in\{1,\dots,n:j\neq\ell,V_j=V_\ell\}}k(\alpha_j,\gamma_j,\gamma_\ell)t+O(e^{-R^2(\sin\gamma)^2/(32t)}),\end{aligned} \tag{11}$$

where $g:(0,2\pi)\to\mathbb{R}$ is given by

$$g(\beta)=\begin{cases}\frac{1}{\pi}+\left(1-\frac{\beta}{\pi}\right)\cot\beta, & \beta\in(0,\pi)\cup(\pi,2\pi);\\ 0, & \beta=\pi,\end{cases} \tag{12}$$

and $k : (0,\pi) \times (0,2\pi) \times (0,2\pi) \to \mathbb{R}$ *is given by*

$$
\begin{aligned}
k(\alpha,\theta,\sigma) &= \frac{1}{2\pi}\left(-(\sigma+\theta+\alpha-\pi)\cot(\sigma+\theta+\alpha) - (\alpha-\pi)\cot\alpha\right) \\
&\quad + \frac{1}{2\pi}\left((\sigma+\alpha-\pi)\cot(\sigma+\alpha) + (\theta+\alpha-\pi)\cot(\theta+\alpha)\right),
\end{aligned} \tag{13}
$$

for $\sigma+\theta+\alpha \neq \pi$, $\alpha \neq \pi$, $\sigma+\alpha \neq \pi$, $\theta+\alpha \neq \pi$, *where* α *denotes the smallest angle between* W_θ *and* W_σ. *In any of the remaining cases, such as* $\alpha = \pi$, *we define* $k(\alpha,\theta,\sigma)$ *by taking appropriate limits using l'Hôpital's rule.*

The terms which involve the area and perimeter are as expected and agree with those in (6). We see that the heat content has a non-trivial dependence on the interior angles of the polygonal boundary.

We observe that $\beta \mapsto g(\beta)$ is continuous on $(0,2\pi)$, decreasing on $(0,\pi]$ and symmetric with respect to π. That is

$$
g(\beta) = g(2\pi-\beta),\ 0<\beta<2\pi. \tag{14}
$$

By (12) and (14), we conclude that g is non-negative. We remark that $k(\alpha,\theta,\sigma)$ is symmetric with respect to θ and σ and that $k(\alpha,\theta,\sigma) = k(2\pi-\theta-\sigma-\alpha,\theta,\sigma)$. By Lemma 10, Section 3, it follows that k is non-negative.

In addition, if D is a regular n-gon in $\mathbb{R}^2$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \left(\frac{n-2}{n}\right)\pi$, and the angular contribution to the heat content is

$$
\frac{n}{\pi} + 2\cot\left(\left(\frac{n-2}{n}\right)\pi\right) = O\left(\frac{1}{n}\right),\ n\to\infty.
$$

We observe that the coefficient of $\int_{\partial\Omega} L_{aa}t$ in the expansion of the heat content of a compact domain Ω with smooth boundary $\partial\Omega$ is also equal to 0, see Theorem 1.6 in [4]. Here L_{aa} is the trace of the second fundamental form when $\partial\Omega$ is oriented with a smooth inward-pointing unit normal vector field.

The expansion for the heat content of a polygon with Dirichlet 0 boundary conditions was obtained in [8]. There it was shown that if v_D solves the heat equation $\Delta v = \frac{\partial v}{\partial t},\quad x \in D$, $t>0$ with $\lim_{t\downarrow 0} v(x;t) = 1,\quad x\in D$ and satisfies a Dirichlet boundary condition $\lim_{x\to x_0} v(x;t) = 0$ for any $x_0 \in \partial D$ then

$$
\int_D dx\, v_D(x;t) = |D| - 2\mathcal{P}(D)\frac{t^{1/2}}{\sqrt{\pi}} + \sum_{j=1}^{n} c(\gamma_j)t + O(e^{-R^2(\sin(\gamma/2))^2/(32t)}), \tag{15}
$$

where

$$
c(\beta) = \int_0^\infty \frac{4\sinh((\pi-\beta)x)}{(\sinh(\pi x))(\cosh(\beta x))}dx.
$$

We note that both (11) and (15) have angular contributions which are additive. However, in Theorem 1 there is an additional term in the case where vertices have multiplicity larger than 1. No such term is present in (15) since sectors based at the same vertex do not feel each other's presence due to the Dirichlet 0 boundary condition.

The strategy to prove (15) is inspired by [14], and relies on some model computations. We use an analogous strategy to prove (11). For points $x \in D$ close to a vertex, say $x \in B_j(r)$ for some $j \in \{1,2,\cdots,n\}$, $r>0$, u_D is approximated by u_{W_j}. For points $x\in D$ which have a distance at least δ to ∂D, for some $\delta>0$, u_D is approximated by 1. For the remaining points in D, u_D is approximated by u_H, where H is the half-plane which contains D and whose boundary contains the edge of ∂D nearest to x. As was the case in [8], the model computations involving the infinite wedge W_j are the most difficult to carry out. However, in contrast to the case with Dirichlet 0 boundary conditions, we must also consider the contribution to the heat content from a vertex which belongs to the boundaries of more than one wedge. We deal with these computations in Lemma 9 and Lemma 10 in Section 3. In Section 4 we carry out the half-plane computations.

It is quite remarkable that, in contrast to the smooth case (5), the asymptotic expansion in half powers of t of $H_D(t)$ in Theorem 1 terminates after the term of order t, leaving an exponentially small remainder as $t\downarrow 0$. This agrees with the fact that there are no further locally computable invariants of D and ∂D available from which non-trivial quantities could be built. A similar phenomenon has been observed for the asymptotic expansion

of the heat trace. See for example [7]. The precise form of the exponential remainder remains an open problem. The extension of the results in this paper to general polyhedra in $\mathbb{R}^3$ is another challenge beyond the scope of this paper.

However, in Section 6, we use Theorem 1 to compute the heat content of a fractal polyhedron which is constructed as follows. Let $Q_0 \subset \mathbb{R}^3$ be an open cube of side-length 1. Let $0 < s < 1$. Attach a regular open cube $Q_{1,i}$ of side-length s to the centre $c_{1,i}, i = 1, \dots, 6$, of each face of ∂Q_0, and such that all the faces are pairwise-parallel. Now proceed by induction. For $j = 2, 3, \dots$, attach $N(j) = 6 \cdot 5^{j-1}$ open cubes $Q_{j,1}, \dots, Q_{j,N(j)}$, of side-length s^j to the centres of the boundary faces of the cubes $Q_{j-1,1}, \dots, Q_{j-1,N(j-1)}$, again with pairwise-parallel faces. We define the fractal polyhedron D_s as

$$D_s = \text{interior}\left\{\overline{Q_0 \cup \left[\bigcup_{j\ge1}\bigcup_{1\le i\le N(j)} Q_{j,i}\right]}\right\}$$

for $0 < s < \sqrt{2} - 1$ (see Figure 1). We note that for this range of s, no cubes in the construction of D_s overlap. For the two-dimensional construction, see Theorem 4 in [6]. In that paper, the critical value $s = \sqrt{2} - 1$, where the squares just touch, is allowed. This is due to the fact that the Dirichlet 0 boundary conditions guarantee the independence of the heat flow in these touching squares. In this paper, we do not impose Dirichlet 0 boundary conditions on ∂D_s, and if the cubes touch, then this could give rise to an extra term. For this reason, we only allow $0 < s < \sqrt{2} - 1$.

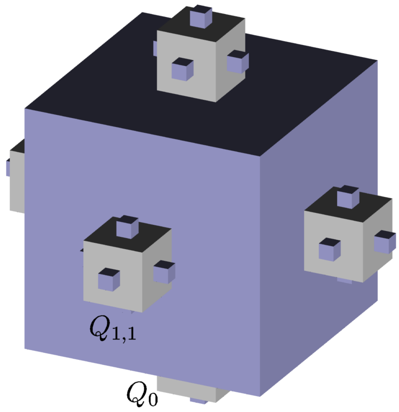

FIGURE 1. The first two generations of D_s with $s = \frac{1}{4}$.

We have that

$$|D_s| = \frac{1+s^3}{1-5s^3},$$

and that the two-dimensional Hausdorff measure of the boundary is given by

$$\mathcal{H}^2(\partial D_s) = 6\left(\frac{1-s^2}{1-5s^2}\right).$$

Both quantities are finite for $0 < s < \sqrt{2} - 1$. Moreover, the total length of the edges of ∂D_s is finite if and only if $0 < s < \frac{1}{5}$ and equals $12\left(\frac{1+s}{1-5s}\right)$. In addition to Theorem 1, in order to compute the heat content of D_s, we require the heat content of a sector of angle π in an infinite wedge of angle $\frac{3\pi}{2}$ where this sector and wedge share one common edge and vertex. This will be computed in Section 5. In Section 6, we prove the following theorems which give the asymptotic expansion for the heat content of D_s as $t \downarrow 0$.

Theorem 2. *Let* $d = \frac{3}{2} + \frac{1}{2}\frac{\log 5}{\log s}$. *Fix* $0 < s < \sqrt{2} - 1$, $s \neq \frac{1}{5}$. *There exists a periodic, continuous function* $p_s : \mathbb{R} \to \mathbb{R}$ *with period* $\log(s^{-2})$ *such that*

$$H_{D_s}(t) = \frac{1+s^3}{1-5s^3} - 6\left(\frac{1-s^2}{1-5s^2}\right)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12}{\pi}\left(\frac{1+s}{1-5s}\right)t + 6p_s(\log t)t^d + O(t^{3/2}\left(\log(t^{-1})\right)^{3/2}),\ t \downarrow 0. \tag{16}$$

It is easy to see that if we write $d = \frac{m-d_s}{2}$, $m = 3$, in Theorem 2 then $d_s = \frac{\log 5}{\log(s^{-1})}$ is the interior Minkowski dimension of the vertices of ∂D_s for $0 < s < \frac{1}{5}$, whereas for $\frac{1}{5} < s < \sqrt{2} - 1$, it is the interior Minkowski dimension of the edges of ∂D_s. Below we state the corresponding result for the critical case $s = \frac{1}{5}$.

Theorem 3. *For* $s = \frac{1}{5}$, *there exists a periodic, continuous function* $p_{\frac{1}{5}} : \mathbb{R} \to \mathbb{R}$ *with period* $\log 25$ *such that,*

$$H_{D_{\frac{1}{5}}}(t) = \frac{21}{20} - \frac{36}{5}\frac{t^{1/2}}{\sqrt{\pi}} + \frac{132}{5\pi}t - \frac{36}{5\pi\log 5}t\log t + 6p_{\frac{1}{5}}(\log t)t + O(t^{3/2}\left(\log(t^{-1})\right)^{3/2}),\ t \downarrow 0. \tag{17}$$

In Section 2 below we introduce some further notation and state and prove several lemmas.

2. Additional notation and lemmas.

Let D be as given in Theorem 1. In the proofs of Lemma 4 to Lemma 8 we use a variant of Kac's principle of not feeling the boundary to reduce the computation of the heat content of D to a collection of model computations. For $r > 0$, $\delta > 0$, $x \in D$, $A \subset \mathbb{R}^2$ we define

$$d(x, A) = \inf\{|x - z| : z \in A\}, \tag{18}$$

and

$$C(\delta, r) = \left\{x \in D\ :\ d(x, \partial D) < \delta,\ x \notin \bigcup_{k=1}^{n} B_k(r)\right\},$$

and

$$D(\delta, r) = \left\{x \in D\ :\ x \notin \bigcup_{k=1}^{n} B_k(r),\ x \notin C(\delta, r)\right\}.$$

Choose R as in (10) and let $\delta = \frac{R}{2}|\sin\gamma|$.

Lemma 4.

$$\int_{D(\delta,R)} dx \int_D dy\, p(x, y; t) = |D(\delta, R)| + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0.$$

Proof. By [3, Proposition 9(i)], we have that

$$1 - 2e^{-\delta(x)^2/(8t)} \leq \int_D dy\, p(x, y; t) \leq 1,$$

where $\delta(x) = \min\{|x - y|\ :\ y \in \mathbb{R}^2 - D\}$. Since $x \in D(\delta, R)$, $\delta(x) \geq \delta$. Hence

$$|D(\delta, R)| - 2|D(\delta, R)|e^{-\delta^2/(8t)} \leq \int_{D(\delta,R)} dx \int_D dy\, p(x, y; t) \leq |D(\delta, R)|$$

as required. □

We partition the region $D - D(\frac{R}{2}|\sin\gamma|, R)$ into n sectors, $B_i(R)$, of radius R, n rectangles, S_γ, of height $\frac{R}{2}|\sin\gamma|$ and $2n$ cusps of height $\frac{R}{2}|\sin\gamma|$. The contributions to $H_D(t)$ from these regions will be computed in Sections 3, 4.1 and 4.2 respectively. Each sector has two neighbouring cusps. Each cusp is adjacent to a rectangle and a sector. The corresponding m-dimensional result to [6, Lemma 7] is the following.

Lemma 5. *Let $\tilde{D}, F, G$ be non-empty, open subsets of $\mathbb{R}^m, m \geq 2$ such that $\tilde{D} \cap F \neq \emptyset$ and $G \subset \tilde{D} \cap F$. Let E be a bounded, measurable subset of G. Then*

$$\int_E dx \int_{\tilde{D}} dy\, p(x,y;t) = \int_E dx \int_F dy\, p(x,y;t) + O(e^{-\epsilon^2/(8t)}),\ t \downarrow 0,$$

where

$$\epsilon = \inf\{|x-y| : x \in E, y \in \overline{(\tilde{D} \cup F) \cap \partial G}\}.$$

Proof. We write

$$\begin{aligned}
&\int_E dx \int_{\tilde{D}} dy\, p(x,y;t) \\
&= \int_E dx \int_{\tilde{D}\cap F} dy\, p(x,y;t) + \int_E dx \int_{\tilde{D}\cap F^c} dy\, p(x,y;t) \\
&= \int_E dx \int_F dy\, p(x,y;t) - \int_E dx \int_{\tilde{D}^c\cap F} dy\, p(x,y;t) + \int_E dx \int_{\tilde{D}\cap F^c} dy\, p(x,y;t). \qquad (19)
\end{aligned}$$

By (19), we have that

$$\begin{aligned}
\int_E dx \int_{\tilde{D}} dy\, p(x,y;t) &\geq \int_E dx \int_F dy\, p(x,y;t) - \int_E dx \int_{\tilde{D}^c\cap F} dy\, p(x,y;t) \\
&\geq \int_E dx \int_F dy\, p(x,y;t) - 2^{m/2}|E|e^{-\epsilon^2/(8t)},
\end{aligned}$$

and

$$\begin{aligned}
\int_E dx \int_{\tilde{D}} dy\, p(x,y;t) &\leq \int_E dx \int_F dy\, p(x,y;t) + \int_E dx \int_{\tilde{D}\cap F^c} dy\, p(x,y;t) \\
&\leq \int_E dx \int_F dy\, p(x,y;t) + 2^{m/2}|E|e^{-\epsilon^2/(8t)}.
\end{aligned}$$

This completes the proof. □

Lemma 6. *Let $i \in \{1, \cdots, n\}$ and $k \in \mathbb{N}$ such that $i+k \leq n$. Suppose $\gamma_i, \gamma_{i+1}, \cdots, \gamma_{i+k}$ are interior angles of ∂D which are supported by edges which meet at the same vertex $V_i = V_{i+1} = \cdots = V_{i+k}$. Then*

(20)

$$\begin{aligned}
&\int_{\cup_{j=i}^{i+k} B_j(R)} dx \int_D dy\, p(x,y;t) \\
&= \sum_{j=i}^{i+k} \int_{B_j(R)} dx \int_{W_j} dy\, p(x,y;t) + \sum_{j\neq \ell, j,\ell=i}^{i+k} \int_{W_j} dx \int_{W_\ell} dy\, p(x,y;t) + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0.
\end{aligned}$$

Proof. By Lemma 5 with $\tilde{D} = D$, $F = \cup_{j=i}^{i+k} W_j$, $E = \cup_{j=i}^{i+k} B_j(R)$ and $G = \{z \in D : d(z, \cup_{j=i}^{i+k} B_j(R)) < \frac{R}{2}|\sin\gamma|\}$, we have that

$$\begin{aligned}
&\int_{\cup_{j=i}^{i+k} B_j(R)} dx \int_D dy\, p(x,y;t) \\
&= \int_{\cup_{j=i}^{i+k} B_j(R)} dx \int_{\cup_{j=i}^{i+k} W_j} dy\, p(x,y;t) + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0. \qquad (21)
\end{aligned}$$

We also have that

$$\begin{aligned}
&\int_{\cup_{j=i}^{i+k} B_j(R)} dx \int_{\cup_{j=i}^{i+k} W_j} dy\, p(x,y;t) \\
&= \sum_{j=i}^{i+k} \int_{B_j(R)} dx \int_{W_j} dy\, p(x,y;t) + \sum_{j\neq \ell, j,\ell=i}^{i+k} \int_{W_j} dx \int_{W_\ell} dy\, p(x,y;t) \\
&\quad + O(e^{-R^2/(8t)}),\ t \downarrow 0. \qquad (22)
\end{aligned}$$

Combining (21) and (22) gives (20). □

Lemma 7. *Let $H \subset \mathbb{R}^2$ denote the half-plane such that $\emptyset \neq \partial S_\gamma \cap \partial D \subset \partial H$ and $S_\gamma \subset H$. Then*

$$\int_{S_\gamma} dx \int_D dy\, p(x,y;t) = \int_{S_\gamma} dx \int_H dy\, p(x,y;t) + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0.$$

Proof. Using Lemma 5 with $\tilde{D} = D$, $F = H$, $E = S_\gamma$ and $G = \{z \in D : d(z, S_\gamma) < \frac{R}{2}|\sin\gamma|\}$, the result follows. □

Lemma 8. *Let C_i denote the cusp which is adjacent to S_γ and $B_i(R)$. Let H be the half-plane as in Lemma 7. Then*

$$\int_{C_i} dx \int_D dy\, p(x,y;t) = \int_{C_i} dx \int_H dy\, p(x,y;t) + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0.$$

Proof. Using Lemma 5 with $\tilde{D} = D$, $F = H$, $E = C_i$ and $G = \{z \in D : d(z, C_i) < \frac{R}{2}|\sin\gamma|\}$, the result follows. □

3. The contribution to the heat content from points close to a vertex.

In this section we approximate u_D by u_{W_j}. We then compute the contribution to the heat content of D from a sector, $B_j(R)$ with corresponding angle $\gamma_j = \beta$. We also compute the contribution to the heat content of D from two disjoint wedges whose boundaries intersect in a vertex of ∂D.

Firstly, we define

$$\mathcal{V}_\beta(t;R) = \int_{B_j(R)} dx \int_{W_j} dy\, p(x,y;t). \tag{23}$$

Lemma 9. *For $\beta \in (0,\pi) \cup (\pi, 2\pi)$,*

$$\begin{aligned}\mathcal{V}_\beta(t;R) = {}& \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}} t^{1/2} + \left(\frac{1}{\pi} + \left(1 - \frac{\beta}{\pi}\right)\cot\beta\right) t \\ & - (4\pi t)^{-1/2} \int_0^{R|\sin\beta|} dx \left(-2Rx + R^2 \arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2 - x^2}\right) e^{-x^2/(4t)} \\ & + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t \downarrow 0.\end{aligned}$$

Proof. Changing to polar coordinates with $x = (r_1\cos\theta_1, r_1\sin\theta_1)$ and $y = (r_2\cos\theta_2, r_2\sin\theta_2)$ in (23) gives that

$$\begin{aligned}&\mathcal{V}_\beta(t;R) \\ &= (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)},\end{aligned}$$

where $A = \cos(\theta_1 - \theta_2)$. The change of variable $r_2 - r_1 A = \rho$ gives that

$$\begin{aligned}\mathcal{V}_\beta(t;R) &= (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_2 - r_1 A)^2/(4t) - r_1^2(1-A^2)/(4t)} \\ &= (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R r\, dr \int_{-Ar}^\infty d\rho\, (\rho + Ar) e^{-\rho^2/(4t) - r^2(1-A^2)/(4t)} \\ &= I_1 + I_2.\end{aligned}$$

We have that

$$\begin{aligned}I_1 &= (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R r\, dr \int_{-Ar}^\infty d\rho\, \rho e^{-\rho^2/(4t) - r^2(1-A^2)/(4t)} \\ &= \frac{\beta^2}{\pi} t (1 - e^{-R^2/(4t)}),\end{aligned} \tag{24}$$

and

$$\begin{aligned}
I_2 &= (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,Ar^2\int_{-Ar}^\infty d\rho\, e^{-\rho^2/(4t)-r^2(1-A^2)/(4t)}\\
&= (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,Ar^2 e^{-r^2(1-A^2)/(4t)}\int_0^\infty d\rho\, e^{-\rho^2/(4t)}\\
&\quad + (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,A^2r^2 e^{-r^2(1-A^2)/(4t)}\int_0^r d\rho\, e^{-A^2\rho^2/(4t)}\\
&= (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,Ar^2 e^{-r^2(1-A^2)/(4t)}\int_0^\infty d\rho\, e^{-\rho^2/(4t)}\\
&\quad + (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,A^2r^2 e^{-r^2(1-A^2)/(4t)}\int_0^\infty d\rho\, e^{-A^2\rho^2/(4t)}\\
&\quad - (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,A^2r^2 e^{-r^2(1-A^2)/(4t)}\int_r^\infty d\rho\, e^{-A^2\rho^2/(4t)}\\
&= I_3 + I_4 + I_5.
\end{aligned}$$

Via the change of variables $\theta_1-\theta_2=-\eta$ and integrating by parts with respect to θ, we obtain

$$\begin{aligned}
I_3 &= (4\pi t)^{-1}\int_0^\beta d\theta_1\int_0^\beta d\theta_2\int_0^R dr\,Ar^2 e^{-r^2(1-A^2)/(4t)}\int_0^\infty d\rho\, e^{-\rho^2/(4t)}\\
&= (4\pi t)^{-1/2}\int_0^\beta d\theta_1\int_{\theta_1}^\beta d\theta_2\int_0^R dr\,r^2\cos(\theta_1-\theta_2)\, e^{-r^2(\sin(\theta_1-\theta_2)^2)/(4t)}\\
&= (4\pi t)^{-1/2}\int_0^\beta d\theta_1\int_0^{\beta-\theta_1} d\eta\int_0^R dr\,r^2\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}\\
&= (4\pi t)^{-1/2}\int_0^\beta d\theta\int_0^{\theta} d\eta\int_0^R dr\,r^2\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}\\
&= \beta(4\pi t)^{-1/2}\int_0^\beta d\eta\int_0^R dr\,r^2\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}\\
&\quad - (4\pi t)^{-1/2}\int_0^\beta d\theta\,\theta\int_0^R dr\,r^2\cos\theta\, e^{-r^2(\sin\theta)^2/(4t)}.
\end{aligned}\tag{25}$$

Similarly,

$$\begin{aligned}
I_4 &= \beta(4\pi t)^{-1/2}\int_0^\beta d\eta\int_0^R dr\,r^2|\cos\eta|\, e^{-r^2(\sin\eta)^2/(4t)}\\
&\quad - (4\pi t)^{-1/2}\int_0^\beta d\theta\,\theta\int_0^R dr\,r^2|\cos\theta|\, e^{-r^2(\sin\theta)^2/(4t)}.
\end{aligned}\tag{26}$$

We first compute I_3+I_4 and then we deal with I_5. By (25) and (26), we have that

$$I_3+I_4 = 2(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_B d\eta\,(\beta-\eta)\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)},\tag{27}$$

where $B=[0,\beta]\cap([0,\frac{\pi}{2}]\cup[\frac{3\pi}{2},2\pi])$. First suppose $\beta\in[0,\frac{\pi}{2}]$, then by (27) we have that

$$\begin{aligned}
&I_3+I_4\\
&= 2(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_0^\beta d\eta\,(\beta-\eta)\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}\\
&= 2(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_0^{\sin\beta} d\psi\,(\beta-\arcsin\psi)\, e^{-r^2\psi^2/(4t)}\\
&= 2\beta(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_0^{\sin\beta} d\psi\, e^{-r^2\psi^2/(4t)} - 2(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_0^{\sin\beta} d\psi\,\psi e^{-r^2\psi^2/(4t)}\\
&\quad - 2(4\pi t)^{-1/2}\int_0^R dr\,r^2\int_0^{\sin\beta} d\psi\,(\arcsin\psi-\psi) e^{-r^2\psi^2/(4t)}
\end{aligned}$$

$$
\begin{aligned}
&= 2\beta(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^\infty d\psi\, e^{-r^2\psi^2/(4t)} - 2\beta(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_{\sin\beta}^\infty d\psi\, e^{-r^2\psi^2/(4t)} \\
&\quad + 4t(4\pi t)^{-1/2}\int_0^R dr\, (e^{-r^2(\sin\beta)^2/(4t)} - 1) \\
&\quad - 2(4\pi t)^{-1/2}\int_0^{\sin\beta} d\psi\, (\arcsin\psi - \psi)\int_0^R dr\, r^2 e^{-r^2\psi^2/(4t)} \\
&= \frac{\beta R^2}{2} - 2\beta(4\pi t)^{-1/2}\int_{\sin\beta}^\infty d\psi \int_0^\infty dr\, r^2\, e^{-r^2\psi^2/(4t)} - \frac{2R}{\sqrt{\pi}}t^{1/2} + \frac{2}{\sin\beta}t \\
&\quad - 2(4\pi t)^{-1/2}\int_0^{\sin\beta} d\psi\, (\arcsin\psi - \psi)\int_0^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} \\
&\quad + 2(4\pi t)^{-1/2}\int_0^{\sin\beta} d\psi\, (\arcsin\psi - \psi)\int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)}) \\
&= \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}}t^{1/2} - 2\beta t\int_{\sin\beta}^\infty \frac{d\psi}{\psi^3} + \frac{2}{\sin\beta}t - 2t\int_0^{\sin\beta} d\psi\, \left(\frac{\arcsin\psi - \psi}{\psi^3}\right) \\
&\quad + 2(4\pi t)^{-1/2}\int_0^{\sin\beta} d\psi\, (\arcsin\psi - \psi)\int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)})
\end{aligned}
$$

(28)

$$
\begin{aligned}
&= \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}}t^{1/2} + (\cot\beta)t + 2(4\pi t)^{-1/2}\int_0^{\sin\beta} d\psi\, (\arcsin\psi - \psi)\int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} \\
&\quad + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t\downarrow 0.
\end{aligned}
$$

Similarly, for $\beta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we have that

$$
\begin{aligned}
&I_3 + I_4 \\
&= 2(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^{\frac{\pi}{2}} d\eta\, (\beta - \eta)\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)} \\
&= 2(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^1 d\psi\, (\beta - \arcsin\psi)\, e^{-r^2\psi^2/(4t)} \\
&= 2\beta(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^1 d\psi\, e^{-r^2\psi^2/(4t)} - 2(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^1 d\psi\, \psi e^{-r^2\psi^2/(4t)} \\
&\quad - 2(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^1 d\psi\, (\arcsin\psi - \psi) e^{-r^2\psi^2/(4t)} \\
&= 2\beta(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_0^\infty d\psi\, e^{-r^2\psi^2/(4t)} - 2\beta(4\pi t)^{-1/2}\int_0^R dr\, r^2 \int_1^\infty d\psi\, e^{-r^2\psi^2/(4t)} \\
&\quad + 4t(4\pi t)^{-1/2}\int_0^R dr\, (e^{-r^2/(4t)} - 1) \\
&\quad - 2(4\pi t)^{-1/2}\int_0^1 d\psi\, (\arcsin\psi - \psi)\int_0^R dr\, r^2 e^{-r^2\psi^2/(4t)} \\
&= \frac{\beta R^2}{2} - 2\beta t\int_1^\infty \frac{d\psi}{\psi^3} - \frac{2R}{\sqrt{\pi}}t^{1/2} + 2t - 2t\int_0^1 d\psi\, \left(\frac{\arcsin\psi - \psi}{\psi^3}\right) \\
&\quad + 2(4\pi t)^{-1/2}\int_0^1 d\psi\, (\arcsin\psi - \psi)\int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)})
\end{aligned}
$$

(29)

$$
\begin{aligned}
&= \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}}t^{1/2} + \left(\frac{\pi}{2} - \beta\right)t + 2(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\, (\arcsin\psi - \psi)\int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} \\
&\quad + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t\downarrow 0.
\end{aligned}
$$

In order to have (29) in the same format as (28), we replaced 1 by $|\sin\beta|$ in (29). For $\beta \in [\frac{3\pi}{2}, 2\pi]$, we have that

$$\begin{aligned} I_3 + I_4 &= 2(4\pi t)^{-1/2} \int_0^R dr\, r^2 \int_0^{\frac{\pi}{2}} d\eta\, (\beta-\eta) \cos\eta\, e^{-r^2(\sin\eta)^2/(4t)} \\ &\quad + 2(4\pi t)^{-1/2} \int_0^R dr\, r^2 \int_{\frac{3\pi}{2}}^{\beta} d\eta\, (\beta-\eta)\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)} \\ &= J_1 + J_2. \end{aligned}$$

Now by (29),

(30)

$$J_1 = \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}} t^{1/2} + \left(\frac{\pi}{2} - \beta\right) t + 2(4\pi t)^{-1/2} \int_0^{|\sin\beta|} d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)}$$
$$+ O(te^{-R^2(\sin\beta)^2/(8t)}),\ t \downarrow 0.$$

Via the change of variables $\eta' = 2\pi - \eta$, we also have that

$$\begin{aligned} J_2 &= 2(4\pi t)^{-1/2} \int_{\frac{3\pi}{2}}^{\beta} d\eta\, (\beta-\eta)\cos\eta \int_0^R dr\, r^2\, e^{-r^2(\sin\eta)^2/(4t)} \\ &= 2(4\pi t)^{-1/2} \int_{2\pi-\beta}^{\frac{\pi}{2}} d\eta\, (\beta+\eta-2\pi)\cos\eta \int_0^R dr\, r^2\, e^{-r^2(\sin\eta)^2/(4t)} \\ &= 2t \int_{2\pi-\beta}^{\frac{\pi}{2}} d\eta\, (\beta+\eta-2\pi) \frac{\cos\eta}{(\sin\eta)^3} \\ &\quad - 2(4\pi t)^{-1/2} \int_{2\pi-\beta}^{\frac{\pi}{2}} d\eta\, (\beta+\eta-2\pi)\cos\eta \int_R^\infty dr\, r^2\, e^{-r^2(\sin\eta)^2/(4t)} \end{aligned}$$

$$= \left(\frac{3\pi}{2} - \beta - \cot\beta\right) t + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t \downarrow 0. \tag{31}$$

Hence by (30) and (31), we see that for $\beta \in [\frac{3\pi}{2}, 2\pi]$ and $t \downarrow 0$,

(32)

$$\begin{aligned} I_3 + I_4 &= \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}} t^{1/2} + (2\pi - 2\beta - \cot\beta)t \\ &\quad + 2(4\pi t)^{-1/2} \int_0^{|\sin\beta|} d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)}). \end{aligned}$$

It remains to compute I_5. Via the change of variables $\rho = r\rho'$, we see that

$$\begin{aligned} I_5 &= -(4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr\, A^2 r^2 e^{-r^2(1-A^2)/(4t)} \int_r^\infty d\rho\, e^{-A^2\rho^2/(4t)} \\ &= -(4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr\, A^2 r^3 e^{-r^2(1-A^2)/(4t)} \int_1^\infty d\rho\, e^{-r^2A^2\rho^2/(4t)} \\ &= -(8\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^{R^2} dr\, A^2 r \int_1^\infty d\rho\, e^{-r(A^2\rho^2+1-A^2)/(4t)} \\ &= -(8\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_1^\infty d\rho \int_0^\infty dr\, A^2 r e^{-r(A^2\rho^2+1-A^2)/(4t)} \\ &\quad + (8\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_1^\infty d\rho \int_{R^2}^\infty dr\, A^2 r e^{-r(A^2\rho^2+1-A^2)/(4t)} \end{aligned}$$

$$= -\frac{2t}{\pi} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_1^\infty d\rho \frac{A^2}{(A^2\rho^2+1-A^2)^2} + O(te^{-R^2/(4t)}),\ t \downarrow 0. \tag{33}$$

By 2.173 (1) and 2.172 in [13], we see that

$$\int_1^\infty \frac{d\rho}{(A^2\rho^2+1-A^2)^2} = \frac{-1}{2(1-A^2)} + \frac{1}{2|A|(1-A^2)^{3/2}} \arctan\left(\frac{\sqrt{1-A^2}}{|A|}\right). \tag{34}$$

Therefore by (33), (34) and the change of variables $\theta_2 - \theta_1 = \sigma$, we obtain that

$$\begin{aligned}
&-\frac{2t}{\pi}\int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_1^\infty d\rho \frac{A^2}{(A^2\rho^2+1-A^2)^2}\\
&= -\frac{2t}{\pi}\int_0^\beta d\theta \int_0^\theta d\sigma \left(-(\cot\sigma)^2 + \frac{\cos\sigma}{(\sin\sigma)^3}\arctan(\tan\sigma)\right).
\end{aligned}$$

Thus

$$I_5 = -\frac{2t}{\pi}\int_0^\beta d\theta \int_0^\theta d\sigma \left(-(\cot\sigma)^2 + \frac{\cos\sigma}{(\sin\sigma)^3}\arctan(\tan\sigma)\right) + O(te^{-R^2/(4t)}),\ t\downarrow 0.$$

We note that

$$\arctan(\tan\sigma) = \sigma + U(\sigma),$$

where

$$U(\sigma) = \begin{cases} 0, & \text{if } \sigma\in(0,\frac{\pi}{2});\\ -\pi, & \text{if } \sigma\in(\frac{\pi}{2},\frac{3\pi}{2});\\ -2\pi, & \text{if } \sigma\in(\frac{3\pi}{2},2\pi).\end{cases}$$

Hence it is necessary to compute

$$\begin{aligned}
-\frac{2t}{\pi}\int_0^\beta d\theta \int_0^\theta d\sigma \left(-(\cot\sigma)^2 + \frac{\sigma\cos\sigma}{(\sin\sigma)^3}\right) &= -\frac{2t}{\pi}\int_0^\beta d\theta\left(\frac{\cot\theta}{2} + \theta - \frac{\theta}{2(\sin\theta)^2}\right)\\
&= \left(-\frac{\beta^2}{\pi} - \frac{\beta}{\pi}\cot\beta + \frac{1}{\pi}\right)t.
\end{aligned}$$

If $\beta\in(\frac{\pi}{2},\frac{3\pi}{2})$, then

$$-\frac{2t}{\pi}\int_0^\beta d\theta\int_0^\theta d\sigma\, U(\sigma)\frac{\cos\sigma}{(\sin\sigma)^3} = 2t\int_{\frac{\pi}{2}}^\beta d\theta \int_{\frac{\pi}{2}}^\theta d\sigma\frac{\cos\sigma}{(\sin\sigma)^3} = \left(\beta + \cot\beta - \frac{\pi}{2}\right)t,$$

and if $\beta\in(\frac{3\pi}{2},2\pi)$, then

$$\begin{aligned}
&-\frac{2t}{\pi}\int_0^\beta d\theta\int_0^\theta d\sigma\, U(\sigma)\frac{\cos\sigma}{(\sin\sigma)^3}\\
&= 2t\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta\int_{\frac{\pi}{2}}^\theta d\sigma\frac{\cos\sigma}{(\sin\sigma)^3} + 2t\int_{\frac{3\pi}{2}}^\beta d\theta\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\sigma\frac{\cos\sigma}{(\sin\sigma)^3} + 4t\int_{\frac{3\pi}{2}}^\beta d\theta\int_{\frac{3\pi}{2}}^\theta d\sigma\frac{\cos\sigma}{(\sin\sigma)^3}\\
&= (2\beta + 2\cot\beta - 2\pi)t.
\end{aligned}$$

Hence

$$I_5 = \begin{cases} \left(-\frac{\beta^2}{\pi} - \frac{\beta}{\pi}\cot\beta + \frac{1}{\pi}\right)t, & \text{if } \beta\in(0,\frac{\pi}{2});\\ \left(-\frac{\beta^2}{\pi} + \left(1-\frac{\beta}{\pi}\right)\cot\beta + \beta - \frac{\pi}{2} + \frac{1}{\pi}\right)t, & \text{if } \beta\in(\frac{\pi}{2},\frac{3\pi}{2});\\ \left(-\frac{\beta^2}{\pi} + \left(1-\frac{\beta}{\pi}\right)\cot\beta + \cot\beta + 2\beta - 2\pi + \frac{1}{\pi}\right)t, & \text{if } \beta\in(\frac{3\pi}{2},2\pi);\end{cases} \tag{35}$$

Therefore, by combining (24), (35) and (28), (29), (32) respectively, we obtain that, as $t\downarrow 0$,

$$\begin{aligned}
\mathcal{V}_\beta(t;R) &= \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}}t^{1/2} + \left(\frac{1}{\pi} + \left(1-\frac{\beta}{\pi}\right)\cot\beta\right)t\\
&\quad + 2(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\,(\arcsin\psi - \psi)\int_R^\infty dr\, r^2\, e^{-r^2\psi^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)}).
\end{aligned}$$

Via the change of variables $\rho = r\psi$ and integrating by parts with respect to ψ, we have

$$\begin{aligned}
&2(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\,(\arcsin\psi - \psi)\int_R^\infty dr\, r^2\, e^{-r^2\psi^2/(4t)}\\
&= 2(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\,\left(\frac{\arcsin\psi-\psi}{\psi^3}\right)\int_{R\psi}^\infty d\rho\,\rho^2 e^{-\rho^2/(4t)}\\
&= -(4\pi t)^{-1/2}R^3\int_0^{|\sin\beta|} d\psi\,(\arcsin\psi - \psi)e^{-R^2\psi^2/(4t)}\\
&\quad + (4\pi t)^{-1/2}\int_0^{|\sin\beta|}\frac{d\psi}{\psi^2}\left(\frac{1-\sqrt{1-\psi^2}}{\sqrt{1-\psi^2}}\right)\int_{R\psi}^\infty d\rho\,\rho^2 e^{-\rho^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)})\\
&= -(4\pi t)^{-1/2}R^3\int_0^{|\sin\beta|} d\psi\,(\arcsin\psi - \psi)e^{-R^2\psi^2/(4t)}\\
&\quad + (4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\left(\frac{1}{\sqrt{1-\psi^2}} - \frac{1}{1+\sqrt{1-\psi^2}}\right)\int_{R\psi}^\infty d\rho\,\rho^2 e^{-\rho^2/(4t)}\\
&\quad + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t\downarrow 0.
\end{aligned}\tag{36}$$

Again integrating by parts with respect to ψ, we see that

$$\begin{aligned}
&(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\left(\frac{1}{\sqrt{1-\psi^2}} - \frac{1}{1+\sqrt{1-\psi^2}}\right)\int_{R\psi}^\infty d\rho\,\rho^2 e^{-\rho^2/(4t)}\\
&= (4\pi t)^{-1/2}R^3\int_0^{|\sin\beta|} d\psi\,(-\psi\sqrt{1-\psi^2}+\psi)e^{-R^2\psi^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t\downarrow 0.
\end{aligned}\tag{37}$$

Hence by (36), (37) and the change of variables $\psi = \frac{x}{R}$, we have that

$$\begin{aligned}
&2(4\pi t)^{-1/2}\int_0^{|\sin\beta|} d\psi\,(\arcsin\psi - \psi)\int_R^\infty dr\, r^2\, e^{-r^2\psi^2/(4t)}\\
&= -(4\pi t)^{-1/2}\int_0^{R|\sin\beta|} dx\,\left(R^2\arcsin\left(\frac{x}{R}\right) - Rx\right)e^{-x^2/(4t)}\\
&\quad + (4\pi t)^{-1/2}\int_0^{R|\sin\beta|} dx\,\left(-x\sqrt{R^2-x^2}+Rx\right)e^{-x^2/(4t)} + O(te^{-R^2(\sin\beta)^2/(8t)})\\
&= -(4\pi t)^{-1/2}\int_0^{R|\sin\beta|} dx\,\left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right)e^{-x^2/(4t)}\\
&\quad + O(te^{-R^2(\sin\beta)^2/(8t)}),\ t\downarrow 0.
\end{aligned}\tag{38}$$

This completes the proof of Lemma 9. □

In Section 4.3, we deal with the remaining integral in (38).

Lemma 10. *Let W_1, W_2 be two disjoint wedges in $\mathbb{R}^2$ with corresponding angles γ_1, γ_2 respectively such that $\partial W_1 \cap \partial W_2 = \{V_i\}$ for some $i \in \{1, \dots, n\}$. Let α denote the angle between W_1 and W_2 such that $0 < \alpha \le \pi$. Then*

$$\int_{W_1} dx \int_{W_2} dy\, p(x,y;t) = k(\alpha,\gamma_1,\gamma_2)t,$$

where $k(\alpha,\gamma_1,\gamma_2)$ is as defined in (13).

Proof. By changing coordinates to polar coordinates, as in the proof of Lemma 9, we have that

$$
\begin{aligned}
&\int_{W_1} dx \int_{W_2} dy\, p(x,y;t) \\
&= (4\pi t)^{-1} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha}^{\gamma_1+\alpha+\gamma_2} d\theta_2 \int_0^\infty dr_1 \int_0^\infty dr_2 (r_1 r_2)\, e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2\cos(\theta_1-\theta_2)/(4t)} \\
&= \frac{4t}{\pi} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha}^{\gamma_1+\alpha+\gamma_2} d\theta_2 \int_0^\infty dr_1 \int_0^\infty dr_2 (r_1 r_2)\, e^{-(r_1^2+r_2^2)+2r_1r_2\cos(\theta_1-\theta_2)} \\
&= \frac{4t}{\pi} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha}^{\gamma_1+\alpha+\gamma_2} d\theta_2 \int_0^\infty dr_1\, r_1^3 \int_0^\infty d\rho\, \rho\, e^{-r_1^2(1+\rho^2-2\rho\cos(\theta_1-\theta_2))} \\
&= \frac{2t}{\pi} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha}^{\gamma_1+\alpha+\gamma_2} d\theta_2 \int_0^\infty d\rho\, \rho \int_0^\infty dr\, r\, e^{-r(1+\rho^2-2\rho\cos(\theta_1-\theta_2))} \\
&= \frac{2t}{\pi} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha}^{\gamma_1+\alpha+\gamma_2} d\theta_2 \int_0^\infty d\rho \frac{\rho}{(1+\rho^2-2\rho\cos(\theta_1-\theta_2))^2} \\
&= \frac{2t}{\pi} \int_0^{\gamma_1} d\theta_1 \int_{\gamma_1+\alpha-\theta_1}^{\gamma_1+\alpha+\gamma_2-\theta_1} d\varphi \int_0^\infty d\rho \frac{\rho}{(1+\rho^2-2\rho\cos\varphi)^2} \\
&= \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \int_0^\infty d\rho \frac{\rho}{(1+\rho^2-2\rho\cos\varphi)^2} \\
&= \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \int_0^\infty d\rho \left(\frac{\rho-\cos\varphi}{(1+\rho^2-2\rho\cos\varphi)^2} + \frac{\cos\varphi}{(1+\rho^2-2\rho\cos\varphi)^2} \right) \\
&= \frac{\gamma_1\gamma_2}{\pi} t + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \int_{-\cos\varphi}^\infty d\rho \frac{\cos\varphi}{(\rho^2+(\sin\varphi)^2)^2} \\
&= \frac{\gamma_1\gamma_2}{\pi} t + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \frac{\cos\varphi}{|\sin\varphi|^3} \int_0^\infty \frac{d\rho}{(\rho^2+1)^2} \\
&\quad + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \frac{|\cos\varphi|}{|\sin\varphi|^3} \int_0^{|\cot\varphi|} \frac{d\rho}{(\rho^2+1)^2} \\
&= \frac{\gamma_1\gamma_2}{\pi} t + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \frac{\cos\varphi}{|\sin\varphi|^3} \int_0^{\frac{\pi}{2}} d\theta\, (\cos\theta)^2 \\
&\quad + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \frac{|\cos\varphi|}{|\sin\varphi|^3} \int_0^{\arctan(|\cot\varphi|)} d\theta\, (\cos\theta)^2
\end{aligned}
$$

(39)

$$
\begin{aligned}
&= \frac{\gamma_1\gamma_2}{\pi} t + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi\, \frac{\pi}{4} \frac{\cos\varphi}{|\sin\varphi|^3} \\
&\quad + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \left(\frac{\sin(2\arctan(|\cot\varphi|))}{4} + \frac{\arctan(|\cot\varphi|)}{2} \right) \frac{|\cos\varphi|}{|\sin\varphi|^3}.
\end{aligned}
$$

We see that

$$
S(\varphi) := \arctan(|\cot\varphi|) = \begin{cases} |\varphi - \frac{\pi}{2}|, & \text{if } 0 < \varphi \le \pi; \\ |\varphi - \frac{3\pi}{2}|, & \text{if } \pi < \varphi \le 2\pi. \end{cases}
$$

Hence

$$
\frac{\sin(2\arctan(|\cot\varphi|))}{4} \frac{|\cos\varphi|}{|\sin\varphi|^3} = \frac{1}{2} \left(\frac{\cos\varphi}{\sin\varphi} \right)^2.
$$

Therefore (39) becomes

$$
\frac{\gamma_1\gamma_2}{\pi} t + \frac{2t}{\pi} \int_\alpha^{\gamma_1+\alpha} d\sigma \int_\sigma^{\sigma+\gamma_2} d\varphi \left(\frac{\pi}{4} \frac{\cos\varphi}{|\sin\varphi|^3} + \frac{1}{2} \left(\frac{\cos\varphi}{\sin\varphi} \right)^2 + \frac{S(\varphi)|\cos\varphi|}{2|\sin\varphi|^3} \right). \tag{40}
$$

By considering the function

$$
f(\varphi) = \frac{\pi}{4} \frac{\cos\varphi}{|\sin\varphi|^3} + \frac{1}{2} \left(\frac{\cos\varphi}{\sin\varphi} \right)^2 + \frac{S(\varphi)|\cos\varphi|}{2|\sin\varphi|^3}
$$

for φ in each of the four quadrants, we have that

$$\frac{1}{2}\int_{\sigma}^{\sigma+\gamma_2} d\varphi \left((\pi-\varphi)\frac{\cos\varphi}{(\sin\varphi)^3} + \left(\frac{\cos\varphi}{\sin\varphi}\right)^2 \right)$$

is a primitive for $f(\varphi)$ for all $\varphi \in (0, 2\pi)$. Integrating by parts with respect to φ, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\sigma}^{\sigma+\gamma_2} d\varphi \left((\pi-\varphi)\frac{\cos\varphi}{(\sin\varphi)^3} + \left(\frac{\cos\varphi}{\sin\varphi}\right)^2 \right) \\ &= \frac{1}{4}\left(\frac{(\gamma_2+\sigma-\pi)}{(\sin(\gamma_2+\sigma))^2} - \cot(\gamma_2+\sigma) - \frac{(\sigma-\pi)}{(\sin\sigma)^2} + \cot\sigma - 2\gamma_2 \right). \end{aligned} \tag{41}$$

Hence, by (40), (41) and integrating by parts with respect to σ, we have that

$$\begin{aligned} &\int_{W_1} dx \int_{W_2} dy\, p(x,y;t) \\ &= \frac{t}{2\pi}\left(-(\gamma_2+\gamma_1+\alpha-\pi)\cot(\gamma_2+\gamma_1+\alpha) - (\alpha-\pi)\cot\alpha\right) \\ &\quad + \frac{t}{2\pi}\left((\gamma_2+\alpha-\pi)\cot(\gamma_2+\alpha) + (\gamma_1+\alpha-\pi)\cot(\gamma_1+\alpha)\right) \end{aligned}$$

as required. □

4. The contribution to the heat content from points close to an edge.

In this section, we consider points $x \in C(\frac{R}{2}|\sin\gamma|, R)$. We partition this region into n rectangles S_γ and $2n$ cusps C_i, each of height $\frac{R}{2}|\sin\gamma|$. We approximate u_D by u_H, where $H \subset \mathbb{R}^2$ is the half-plane such that $\emptyset \neq \partial S_\gamma \cap \partial D \subset \partial H$ and $S_\gamma \subset H$, as in Lemma 7.

4.1. The contribution to the heat content from a rectangle.

We first compute the contribution to the heat content of D from a rectangle, S_γ, of height $\frac{R}{2}|\sin\gamma|$ and length L, where $L \in \mathbb{R}, L > 0$. We have that

$$\begin{aligned} \int_H dy\, p(x,y;t) &= \int_{-\infty}^{\infty} dy_2 \int_0^{\infty} dy_1 (4\pi t)^{-1} e^{-(x_1-y_1)^2/(4t)-(x_2-y_2)^2/(4t)} \\ &= 1 - \int_{x_1}^{\infty} d\zeta\, (4\pi t)^{-1/2} e^{-\zeta^2/(4t)}. \end{aligned}$$

Let $x_1 \in (0, \frac{R}{2}|\sin\gamma|)$ and $x_2 \in (0, L)$. Then, by integrating by parts with respect to x_1, we obtain

$$\begin{aligned} &\int_{S_\gamma} dx \int_H dy\, p(x,y;t) \\ &= |S_\gamma| - \int_{S_\gamma} dx \int_{x_1}^{\infty} d\zeta\, (4\pi t)^{-1/2} e^{-\zeta^2/(4t)} \\ &= |S_\gamma| - (4\pi t)^{-1/2}\int_0^L dx_2 \int_0^{\frac{R}{2}|\sin\gamma|} dx_1 \int_{x_1}^{\infty} d\zeta\, e^{-\zeta^2/(4t)} \\ &= |S_\gamma| - (4\pi t)^{-1/2}\int_0^L dx_2 \int_0^{\infty} dx_1 \int_{x_1}^{\infty} d\zeta\, e^{-\zeta^2/(4t)} + O(t^{1/2}e^{-R^2(\sin\gamma)^2/(32t)}) \\ &= |S_\gamma| - \frac{L}{\sqrt{\pi}}t^{1/2} + O(t^{1/2}e^{-R^2(\sin\gamma)^2/(32t)}),\ t \downarrow 0. \end{aligned} \tag{42}$$

4.2. The contribution to the heat content from a cusp.

We now compute the contribution to the heat content of D from a cusp. Let C_i denote the cusp which is

adjacent to S_γ and $B_i(R)$. Then

$$\begin{aligned}\int_{C_i} dx \int_H dy\, p(x,y;t) &= \int_0^{\frac{R}{2}|\sin\gamma|} dx\,(R-\sqrt{R^2-x^2}) \int_0^\infty dy\,(4\pi t)^{-1/2} e^{-|x-y|^2/(4t)}\\ &= \int_0^{\frac{R}{2}|\sin\gamma|} dx\,(R-\sqrt{R^2-x^2}) \left(1-\int_x^\infty d\zeta\,(4\pi t)^{-1/2} e^{-\zeta^2/(4t)}\right)\\ &= |C_i| - \int_0^{\frac{R}{2}|\sin\gamma|} dx\,(R-\sqrt{R^2-x^2}) \int_x^\infty d\zeta\,(4\pi t)^{-1/2} e^{-\zeta^2/(4t)}.\end{aligned}$$

Integrating by parts with respect to x, we obtain

$$\begin{aligned}&-\int_0^{\frac{R}{2}|\sin\gamma|} dx\,(R-\sqrt{R^2-x^2}) \int_x^\infty d\zeta\,(4\pi t)^{-1/2} e^{-\zeta^2/(4t)}\\ &= -(4\pi t)^{-1/2} \int_0^{\frac{R}{2}|\sin\gamma|} dx\, \left(Rx - \frac{R^2}{2}\arcsin\left(\frac{x}{R}\right) - \frac{x}{2}\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &\quad + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t\downarrow 0.\end{aligned}$$

Hence the contribution from each cusp is

$$\begin{aligned}&\int_{C_i} dx \int_H dy\, p(x,y;t)\\ &= |C_i| + (4\pi t)^{-1/2} \int_0^{\frac{R}{2}|\sin\gamma|} dx\, \left(-Rx + \frac{R^2}{2}\arcsin\left(\frac{x}{R}\right) + \frac{x}{2}\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &\quad + O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t\downarrow 0.\end{aligned}$$

4.3. **The $O(t^{3/2})$ terms from the cusp and sector contributions.** Finally, we deal with the remaining integrals from the sector and cusp contributions. Each sector has two neighbouring cusps so we are interested in the following integral

$$\begin{aligned}&2(4\pi t)^{-1/2} \int_0^{\frac{R}{2}|\sin\gamma|} dx\, \left(-Rx + \frac{R^2}{2}\arcsin\left(\frac{x}{R}\right) + \frac{x}{2}\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &= (4\pi t)^{-1/2} \int_0^{\frac{R}{2}|\sin\gamma|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}. \qquad (43)\end{aligned}$$

By definition of γ, in (8), we can write the remaining integral from the sector contribution, (38), as

$$\begin{aligned}&-(4\pi t)^{-1/2} \int_0^{R|\sin\beta|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &= -(4\pi t)^{-1/2} \int_0^{R|\sin\gamma|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)} \qquad (44)\\ &\quad + O(e^{-R^2(\sin\gamma)^2/(8t)}),\ t\downarrow 0.\end{aligned}$$

Adding (43) and (44), we obtain

$$\begin{aligned}&(4\pi t)^{-1/2} \int_0^{\frac{R}{2}|\sin\gamma|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &-(4\pi t)^{-1/2} \int_0^{R|\sin\gamma|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &= -(4\pi t)^{-1/2} \int_{\frac{R}{2}|\sin\gamma|}^{R|\sin\gamma|} dx\, \left(-2Rx + R^2\arcsin\left(\frac{x}{R}\right) + x\sqrt{R^2-x^2}\right) e^{-x^2/(4t)}\\ &= O(e^{-R^2(\sin\gamma)^2/(32t)}),\ t\downarrow 0.\end{aligned}$$

This completes the proof of Theorem 1.

5. The heat content of a π-sector in a $\frac{3\pi}{2}$-wedge.

In this section, we compute the heat content of a π-sector in a $\frac{3\pi}{2}$-wedge which share one common edge (and vertex). This is a crucial ingredient in the computation of the heat content of the fractal polyhedron D_s, which was constructed in Section 1, and will be used in Section 6.

Lemma 11. *Let $B_\pi(R) \subset W_{\frac{3\pi}{2}}$ such that $B_\pi(R)$ and $W_{\frac{3\pi}{2}}$ share one common edge (and vertex). Then, for $t \downarrow 0$,*

$$\begin{aligned}\int_{B_\pi(R)} dx \int_{W_{\frac{3\pi}{2}}} dy\, p(x,y;t) &\\ = \frac{\pi R^2}{2} - \frac{R}{\sqrt{\pi}} t^{1/2} + (4\pi t)^{-1/2} \int_0^1 d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} &\\ + O(te^{-R^2/(8t)}).\end{aligned} \tag{45}$$

We remark that the coefficient of t is equal to 0 in this case.

Proof. Similarly to the proof of Lemma 9, the left-hand side of (45) equals

$$\begin{aligned}&(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_0^{\frac{3\pi}{2}} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &= (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &\quad - (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{\frac{3\pi}{2}}^{2\pi} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &=: M_1 + M_2.\end{aligned} \tag{46}$$

Now

$$\begin{aligned}M_1 &= (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &= \frac{\pi R^2}{2},\end{aligned} \tag{47}$$

and letting $r_2 - r_1 A = \rho$, we have that

$$\begin{aligned}M_2 &= -(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{\frac{3\pi}{2}}^{2\pi} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &= -(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1^2+r_2^2)/(4t)+2r_1r_2A/(4t)} \\ &= -(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R r\, dr \int_{-Ar}^\infty d\rho\, (\rho + Ar) e^{-\rho^2/(4t)-r^2(1-A^2)/(4t)} \\ &= -\frac{\pi}{2} t(1 - e^{-R^2/(4t)}) \\ &\quad - (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R dr \int_{-Ar}^\infty d\rho\, Ar^2 e^{-\rho^2/(4t)-r^2(1-A^2)/(4t)}.\end{aligned} \tag{48}$$

We also have that

$$\begin{aligned}&- (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R dr \int_{-Ar}^\infty d\rho\, Ar^2 e^{-\rho^2/(4t)-r^2(1-A^2)/(4t)} \\ &= -(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R dr \int_0^\infty d\rho\, Ar^2 e^{-\rho^2/(4t)-r^2(1-A^2)/(4t)} \\ &\quad - (4\pi t)^{-1} \int_0^\pi d\theta_1 \int_{-\frac{\pi}{2}}^0 d\theta_2 \int_0^R dr \int_0^r d\rho\, A^2 r^2 e^{-A^2\rho^2/(4t)-r^2(1-A^2)/(4t)}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(4\pi t)^{-1/2}}{2}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\, Ar^2 e^{-r^2(1-A^2)/(4t)}\\
&\quad - (4\pi t)^{-1}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\int_0^\infty d\rho\, A^2r^2e^{-A^2\rho^2/(4t)-r^2(1-A^2)/(4t)}\\
&\quad + (4\pi t)^{-1}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\int_r^\infty d\rho\, A^2r^2e^{-A^2\rho^2/(4t)-r^2(1-A^2)/(4t)}\\
&= -\frac{(4\pi t)^{-1/2}}{2}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\,(A+|A|)r^2 e^{-r^2(1-A^2)/(4t)}\\
&\quad + (4\pi t)^{-1}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\int_r^\infty d\rho\, A^2r^2e^{-A^2\rho^2/(4t)-r^2(1-A^2)/(4t)}\\
&=: N_1+N_2. \qquad (49)
\end{aligned}
$$

Now N_1 equals

$$
\begin{aligned}
&-\frac{(4\pi t)^{-1/2}}{2}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\,(\cos(\theta_1-\theta_2)+|\cos(\theta_1-\theta_2)|)r^2e^{-r^2(\sin(\theta_1-\theta_2))^2/(4t)}\\
&= -\frac{(4\pi t)^{-1/2}}{2}\int_0^\pi d\theta_1\int_0^{\frac{\pi}{2}} d\theta_2\int_0^R dr\,(\cos(\theta_1+\theta_2)+|\cos(\theta_1+\theta_2)|)r^2e^{-r^2(\sin(\theta_1+\theta_2))^2/(4t)}\\
&= -\frac{(4\pi t)^{-1/2}}{2}\int_0^\pi d\theta_1\int_{\theta_1}^{\frac{\pi}{2}+\theta_1} d\eta\int_0^R dr\,(\cos\eta+|\cos\eta|)r^2e^{-r^2(\sin\eta)^2/(4t)}\\
&= -(4\pi t)^{-1/2}\int_0^{\frac{\pi}{2}} d\theta_1\int_{\theta_1}^{\frac{\pi}{2}} d\eta\int_0^R dr\, r^2\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}\\
&\quad -\frac{(4\pi t)^{-1/2}}{2}\int_0^{\frac{\pi}{2}} d\theta_1\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta_1} d\eta\int_0^R dr\,(\cos\eta+|\cos\eta|)r^2e^{-r^2(\sin\eta)^2/(4t)}\\
&\quad -\frac{(4\pi t)^{-1/2}}{2}\int_{\frac{\pi}{2}}^{\pi} d\theta_1\int_{\theta_1}^{\frac{\pi}{2}+\theta_1} d\eta\int_0^R dr\,(\cos\eta+|\cos\eta|)r^2e^{-r^2(\sin\eta)^2/(4t)}\\
&= -(4\pi t)^{-1/2}\int_0^{\frac{\pi}{2}} d\theta\int_{\theta}^{\frac{\pi}{2}} d\eta\int_0^R dr\, r^2\cos\eta\, e^{-r^2(\sin\eta)^2/(4t)}
\end{aligned}
$$

(50)

$$
\begin{aligned}
&= -\frac{R}{\sqrt{\pi}}t^{1/2}+\frac{\pi}{4}t+(4\pi t)^{-1/2}\int_0^1 d\psi\,(\arcsin\psi-\psi)\int_R^\infty dr\, r^2e^{-r^2\psi^2/(4t)}\\
&\quad + O(te^{-R^2/(8t)}),\ t\downarrow 0,
\end{aligned}
$$

by integrating by parts with respect to θ. In addition, as for the computation of I_5 (see (33)), we have that

$$
\begin{aligned}
N_2 &= (4\pi t)^{-1}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_0^R dr\int_r^\infty d\rho\, A^2r^2e^{-A^2\rho^2/(4t)-r^2(1-A^2)/(4t)}\\
&= \frac{2t}{\pi}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\int_1^\infty d\rho\frac{A^2}{(A^2\rho^2+1-A^2)^2}+O(te^{-R^2/(4t)})\\
&= \frac{t}{\pi}\int_0^\pi d\theta_1\int_{-\frac{\pi}{2}}^0 d\theta_2\left(-(\cot(\theta_1-\theta_2))^2+\frac{\cos(\theta_1-\theta_2)}{(\sin(\theta_1-\theta_2))^3}\arctan(\tan(\theta_1-\theta_2))\right)+O(te^{-R^2/(4t)})\\
&= \frac{t}{\pi}\int_0^\pi d\theta_1\int_0^{\frac{\pi}{2}} d\theta_2\left(-(\cot(\theta_1+\theta_2))^2+\frac{\cos(\theta_1+\theta_2)}{(\sin(\theta_1+\theta_2))^3}\arctan(\tan(\theta_1+\theta_2))\right)+O(te^{-R^2/(4t)})\\
&= \frac{t}{\pi}\int_0^\pi d\theta_1\int_{\theta_1}^{\frac{\pi}{2}+\theta_1} d\eta\left(-(\cot\eta)^2+\frac{\cos\eta}{(\sin\eta)^3}\arctan(\tan\eta)\right)+O(te^{-R^2/(4t)})\\
&= \frac{t}{\pi}\int_0^\pi d\theta\int_{\theta}^{\frac{\pi}{2}+\theta} d\eta\left(-(\cot\eta)^2+\frac{\cos\eta}{(\sin\eta)^3}\arctan(\tan\eta)\right)+O(te^{-R^2/(4t)}),\ t\downarrow 0.
\end{aligned}
$$

Note that

$$\arctan(\tan\eta) = \begin{cases} \eta & \text{if } \eta \in (0, \frac{\pi}{2}); \\ \eta - \pi & \text{if } \eta \in (\frac{\pi}{2}, \frac{3\pi}{2}). \end{cases}$$

Hence

$$\begin{aligned} N_2 &= \frac{t}{\pi} \int_0^\pi d\theta \int_\theta^{\frac{\pi}{2}+\theta} d\eta \left(-(\cot\eta)^2 + \frac{\cos\eta}{(\sin\eta)^3} \arctan(\tan\eta) \right) + O(te^{-R^2/(4t)}) \\ &= \frac{t}{\pi} \int_0^{\frac{\pi}{2}} d\theta \int_\theta^{\frac{\pi}{2}} d\eta \left(-(\cot\eta)^2 + \frac{\eta\cos\eta}{(\sin\eta)^3} \right) \\ &\quad + \frac{t}{\pi} \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta} d\eta \left(-(\cot\eta)^2 + \frac{(\eta-\pi)\cos\eta}{(\sin\eta)^3} \right) \\ &\quad + \frac{t}{\pi} \int_{\frac{\pi}{2}}^{\pi} d\theta \int_\theta^{\frac{\pi}{2}+\theta} d\eta \left(-(\cot\eta)^2 + \frac{(\eta-\pi)\cos\eta}{(\sin\eta)^3} \right) + O(te^{-R^2/(4t)}) \\ &= \frac{t}{2\pi} + \frac{t}{2\pi} + \frac{t}{\pi}\left(\frac{\pi^2}{4} - 1 \right) + O(te^{-R^2/(4t)}) \\ &= \frac{\pi}{4} t + O(te^{-R^2/(4t)}),\ t \downarrow 0. \end{aligned} \tag{51}$$

Thus, by (48), (49), (50) and (51), we have that, as $t \downarrow 0$,

$$M_2 = -\frac{R}{\sqrt{\pi}} t^{1/2} + (4\pi t)^{-1/2} \int_0^1 d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)}). \tag{52}$$

Therefore, by (46), (47) and (52), as $t \downarrow 0$,

$$\begin{aligned} &(4\pi t)^{-1} \int_0^\pi d\theta_1 \int_0^{\frac{3\pi}{2}} d\theta_2 \int_0^R dr_1 \int_0^\infty dr_2 (r_1 r_2) e^{-(r_1\cos\theta_1 - r_2\cos\theta_2)^2/(4t) - (r_1\sin\theta_1 - r_2\sin\theta_2)^2/(4t)} \\ &= \frac{\pi R^2}{2} - \frac{R}{\sqrt{\pi}} t^{1/2} + (4\pi t)^{-1/2} \int_0^1 d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)}). \end{aligned}$$

□

We note that

$$\begin{aligned} \mathcal{V}_{\frac{3\pi}{2}}(t;R) &= \int_{B_{\frac{3\pi}{2}}(R)} dx \int_{W_{\frac{3\pi}{2}}} dy\, p(x,y;t) \\ &= \int_{B_\pi(R)} dx \int_{W_{\frac{3\pi}{2}}} dy\, p(x,y;t) + \int_{B_{\frac{\pi}{2}}(R)} dx \int_{W_{\frac{3\pi}{2}}} dy\, p(x,y;t). \end{aligned}$$

By Theorem 1 and Lemma 11, this implies that, as $t \downarrow 0$,

$$\begin{aligned} &\int_{B_{\frac{\pi}{2}}(R)} dx \int_{W_{\frac{3\pi}{2}}} dy\, p(x,y;t) \\ &= \frac{\pi R^2}{4} - \frac{R}{\sqrt{\pi}} t^{1/2} + g\left(\frac{3\pi}{2}\right) t \\ &\quad + (4\pi t)^{-1/2} \int_0^1 d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)}) \\ &= \frac{\pi R^2}{4} - \frac{R}{\sqrt{\pi}} t^{1/2} + \frac{t}{\pi} \\ &\quad + (4\pi t)^{-1/2} \int_0^1 d\psi\, (\arcsin\psi - \psi) \int_R^\infty dr\, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)}). \end{aligned} \tag{53}$$

The result of Lemma 11 and formula (53) will be used in Section 6 below.

6. The heat content of the fractal polyhedron D_s.

In this section, we use Theorem 1 to compute the heat content of the fractal polyhedron D_s which was constructed in Section 1. To do this, we adapt the scheme of [6] to the three-dimensional setting below. The key step in [6] was to obtain a renewal equation by making a suitable Ansatz for the heat content. The corresponding Ansatz has been made here in (60) and (61) for $0 < s < \sqrt{2}-1, s \neq \frac{1}{5}$, and in (63) and (65) for $s = \frac{1}{5}$.

In order to derive the required renewal equation, we need to compute the contribution to the heat content $H_{D_s}(t)$ from Q_0 and $Q_{1,1}$. We do this below in Lemma 12 and Lemma 13 respectively. In what follows, for $A \subset \mathbb{R}^3$, $d(x, A)$ is the 3-dimensional analogue of (18). We make the following approximations for $u_{D_s}(x;t)$.

Let $0 < \delta \leq \min\left\{\frac{s^2}{2}, \frac{s(1-s)}{4}\right\}$ and let $x \in D_s$. If $d(x, \partial D_s) \geq \delta$, then we have that

$$|u_{D_s}(x;t) - 1| \leq 2^{3/2} e^{-\delta^2/(8t)},$$

by the principle of not feeling the boundary, [3, Proposition 9(i)]. We define

$$\tilde{F} = \{x \in D_s : d(x, \partial D_s) < \delta, d(x, e) > \delta \text{ for all edges } e \in \partial D_s\}.$$

If $x \in \tilde{F}$, then we have that

$$|u_{D_s}(x;t) - u_H(x;t)| \leq 2^{3/2} e^{-\delta^2/(8t)}, \tag{54}$$

where

$$u_H(x;t) = (4\pi t)^{-1/2} \int_{-d(x,\partial D_s)}^{\infty} d\zeta\, e^{-\zeta^2/(4t)},$$

i.e. H is a half-space whose boundary contains the face of ∂D_s nearest to x. Let

$$\tilde{E} = \{x \in D_s : d(x, e) < \delta \text{ for some edge } e \in \partial D_s, d(x, v) > \delta \text{ for all vertices } v \in \partial D_s\}.$$

If $x \in \tilde{E}$, then we have that

$$|u_{D_s}(x;t) - u_W(x;t)| \leq 2^{3/2} e^{-\delta^2/(8t)}, \tag{55}$$

where W is the infinite wedge $W_{\frac{\pi}{2}}$ for entrant edges and W is the infinite wedge $W_{\frac{3\pi}{2}}$ for re-entrant edges. (See the proof of Lemma 12 for further details). The estimates (54), (55) follow by similar arguments to those given in the proof of Lemma 5 with $\tilde{D} = D_s$, $F = H, W$, $E = \tilde{F}, \tilde{E}$ respectively and $G = \{x \in D_s : d(x, E) < \delta\}$.

It remains to approximate $u_{D_s}(x;t)$ for x near a vertex of $\partial Q_0 \cap \partial D_s$, $\partial Q_{1,1} \cap \partial D_s$ respectively. We only require the contribution to the heat content $H_{D_s}(t)$ from these vertices to derive the required renewal equation. The relevant approximation to make here is via a one-sided infinite cone C_v with vertex $v \in \partial D_s$ such that $\partial C_v \supseteq \{x \in \partial D_s : d(x, v) < \delta\}$. Definition aside, no viable expressions are known for $u_{C_v}(x;t)$ in this 3-dimensional setting. For our purposes, it is sufficient to approximate the neighbourhood of each vertex $v \in (\partial Q_0 \cup \partial Q_{1,1}) \cap \partial D_s$ by a cube S_v. Each cube S_v centred at v has side-length 2δ and is chosen such that the faces of ∂S_v are pairwise-parallel to those of ∂Q_0. We are interested in the contribution to the heat content $H_{D_s}(t)$ from the region $S_v \cap D_s$. There are two cases to consider. Either the vertex v is entrant and $S_v \cap D_s$ is $\frac{1}{8}$ of S_v, or the vertex v is re-entrant and $S_v \cap D_s$ is $\frac{5}{8}$ of S_v. If v is entrant, then the coefficient of $t^{3/2}$ in the expansion for $H_{D_s}(t)$ is equal to $\frac{-1}{\pi^{3/2}}$ by separation of variables. Unfortunately, we were unable to compute the coefficient of $t^{3/2}$ for a re-entrant vertex. However, the contribution to the heat content $H_{D_s}(t)$ from each region $S_v \cap D_s$ is of order: δ^3 from the volume, $\delta^2 t^{1/2}$ from the surface area of the adjoining faces of the vertex, and δt from the length of the adjoining edges of the vertex. Thus, if we choose δ as follows:

$$\delta = 8t^{1/2}(\log(t^{-1}))^{1/2}, \tag{56}$$

then the contribution to the heat content $H_{D_s}(t)$ from each region $S_v \cap D_s$ is $O(t^{3/2}(\log(t^{-1}))^{3/2})$. We note that this choice of δ gives $O(e^{-\delta^2/(8t)}) = O(t^8)$.

Lemma 12. *Let $0 < s < \sqrt{2}-1$. Then*

$$\int_{Q_0} dx\, u_{D_s}(x;t) = 1 - 6(1-s^2)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12}{\pi}t + O(t^{3/2}(\log(t^{-1}))^{3/2}, \ t \downarrow 0.$$

Proof. Partition Q_0 into the following sets.
(i) $\partial Q_0 \cap \partial D_s$ has 32 vertices; $v_i, i = 1, \dots, 32$. At each vertex v_i, consider a cube S_i of side-length 2δ centred at v_i. Let $\tilde{S}_i = S_i \cap Q_0, i = 1, \dots, 32$ and $\tilde{S} = \cup_{i=1}^{32} \tilde{S}_i$.
(ii) $\partial Q_0 \cap \partial D_s$ has 36 edges; $e_j, j = 1, \dots, 36$. Let $\tilde{E}_j = \{x \in Q_0 : d(x, e_j) < \delta, x \notin \tilde{S}\}$.
(iii) $\tilde{F} = \left\{x \in Q_0 : d(x, \partial Q_0 \cap \partial D_s) < \delta, x \notin \left(\tilde{S} \cup \bigcup_{j=1}^{36} \tilde{E}_j\right)\right\}$.
(iv) The interior of Q_0 minus (i), (ii) and (iii); an open polygon P_δ with distance at least δ to ∂D_s.
(v) The remainder, which has measure zero.
The contribution to the heat content from (iv) is $|P_\delta| + O(e^{-\delta^2/(8t)}) = |P_\delta| + O(t^{3/2})$, $t \downarrow 0$.

To compute the contribution from (ii), there are two types of edges to consider. Each edge e_j is the intersection of two faces of ∂D_s. For fixed j, let Π_j denote the plane which is orthogonal to these faces and intersects e_j in exactly one point. Apply Lemma 5 with $\tilde{D} = D_s \cap \Pi_j$, $F = W_{\frac{\pi}{2}}, W_{\frac{3\pi}{2}}$ respectively, $E = \tilde{E}_j \cap \Pi_j$ and $G = \{x \in D_s \cap \Pi_j : d(x, \tilde{E}_j \cap \Pi_j) < \delta\}$ so that either;
(I) the contribution from $\tilde{E}_j \cap \Pi_j$ can be approximated by that from a sector $B_\delta(\frac{\pi}{2})$ in a wedge $W_{\frac{\pi}{2}}$, or
(II) the contribution from $\tilde{E}_j \cap \Pi_j$ can be approximated by that from a sector $B_\delta(\pi)$ in a wedge $W_{\frac{3\pi}{2}}$.

We can use Lemma 9 to deduce that the contribution from the edges of type (I) is

$$\begin{aligned} &12(1-2\delta)\left(|\tilde{E}_j \cap \Pi_j| - 2\delta \frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi} t\right) + O(t^{3/2}) \\ &= 12\left(|\tilde{E}_j| - 2\delta(1-2\delta)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi}(1-2\delta)t\right) + O(t^{3/2}),\ t \downarrow 0. \end{aligned}$$

In addition, Lemma 11 gives that the contribution from the edges of type (II) is

$$\begin{aligned} &24(s-2\delta)\left(|\tilde{E}_j \cap \Pi_j| - \delta \frac{t^{1/2}}{\sqrt{\pi}}\right) + O(t^{3/2}) \\ &= 24\left(|\tilde{E}_j| - \delta(s-2\delta)\frac{t^{1/2}}{\sqrt{\pi}}\right) + O(t^{3/2}),\ t \downarrow 0. \end{aligned}$$

Each cross-section of $\tilde{F}$ is a union of rectangles and cuspidal regions. Thus by Section 4, we have that the contribution from $\tilde{F}$ equals

$$|\tilde{F}| - \mathcal{H}^2(\partial \tilde{F})\frac{t^{1/2}}{\sqrt{\pi}} + O(t^{3/2}),\ t \downarrow 0.$$

For edges of type (I), the sector $\tilde{E}_j \cap \Pi_j$ has two cuspidal neighbours. For edges of type (II), the sector $\tilde{E}_j \cap \Pi_j$ has one cuspidal neighbour but by Lemma 11, the term of order $t^{3/2}$ is half that of Lemma 5. Even though the sector and cusp terms of order $t^{3/2}$ cancel out, the remainder is dominated by the contribution from (i), which is $O(t^{3/2}(\log(t^{-1}))^{3/2})$. This completes the proof of Lemma 12. □

Following the strategy of [6], in order to compute $\int_{D_s - Q_0} dx\, u_{D_s}(x;t)$, we introduce a model solution which approximates u_{D_s} in one of the six components of $D_s - Q_0$. Consider the half-space $H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0\}$ and attach one of the six components of $D_s - Q_0$ to H. The resulting set is

$$H_s = \text{interior}\left\{\overline{H \cup \left[\bigcup_{j \geq 1} \bigcup_{1 \leq i \leq \frac{1}{6}N(j)} Q_{j,i}\right]}\right\}.$$

Let u_{H_s} be the unique solution of (2) and (3) with $D = H_s$. Define

$$E(t) = \int_{H_s - H} dx\, u_{H_s}(x;t).$$

Applying Lemma 5 with $\tilde{D} = D_s$, $F = H_s$, $E = H_s - H$ and $G = \{x \in H_s : d(x, E) < \epsilon\}$, where $\epsilon = \frac{(1+s^2)^{1/2}}{2(1-s)}(1 - 2s - s^2)$, we have

$$\begin{aligned}\int_{D_s - Q_0} dx\, u_{D_s}(x;t) &= 6\int_{H_s - H} dx\, u_{D_s}(x;t)\\ &= 6\int_{H_s - H} dx\, u_{H_s}(x;t) + O(e^{-\epsilon^2/(8t)})\\ &= 6E(t) + O(e^{-\epsilon^2/(8t)}),\ t \downarrow 0. \qquad (57)\end{aligned}$$

In contrast to [6], where the temperature of the boundary is fixed for all $t > 0$, we must account for the fact that $H_s - H$ and $H_s - D_s$ feel each other's presence. The choice of ϵ above is a lower bound for the distance between $H_s - H$ and $H_s - D_s$.

Similarly to Lemma 12, we have;

Lemma 13. *Let $0 < s < \sqrt{2} - 1$. Then*

$$\int_{Q_{1,1}} dx\, u_{H_s}(x;t) = s^3 - 5s^2(1 - s^2)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi}t + \tilde{h}(t),\ t \downarrow 0,$$

where $|\tilde{h}(t)| \le \tilde{C}t^{3/2}(\log(t^{-1}))^{3/2}$ for $t \le s^2$ and some constant $\tilde{C} > 0$.

Proof. The proof of Lemma 13 is analogous to that of Lemma 12. We note that in this case there is an additional type of edge to consider. Namely, the edges where the contribution from $(\tilde{E} \cap Q_{1,1}) \cap \Pi_j$ can be approximated by that from a sector $B_{\frac{\pi}{2}}(\delta)$ in a wedge $W_{\frac{3\pi}{2}}$. There are 4 such edges so the contribution to the heat content is

$$\begin{aligned}&4(s - 2\delta)\left(|(\tilde{E} \cap Q_{1,1}) \cap \Pi_j| - \delta\frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi}t\right) + O(t^{3/2})\\ &= 4\left(|\tilde{E} \cap Q_{1,1}| - \delta(s - 2\delta)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{(s - 2\delta)}{\pi}t\right) + O(t^{3/2}),\ t \downarrow 0,\end{aligned}$$

by (53). This completes the proof of Lemma 13 by our choice of δ, (56). ☐

Below we state and prove the corresponding 3-dimensional result to [6, Proposition 4] for completeness.

Lemma 14. *Fix $0 < s < \sqrt{2} - 1$. Then*

$$E(t) = 5s^3E\left(\frac{t}{s^2}\right) + s^3 - 5s^2(1 - s^2)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi}t + h(t), \qquad (58)$$

where $|h(t)| \le \tilde{C}t^{3/2}(\log(t^{-1}))^{3/2}$ for $t \le s^2$ and some constant $\tilde{C} > 0$.

Proof. Re-write $E(t)$ as

$$E(t) = \int_{Q_{1,1}} dx\, u_{H_s}(x;t) + \int_{H_s - (H \cup Q_{1,1})} dx\, u_{H_s}(x;t).$$

Then $H_s - (H \cup Q_{1,1})$ consists of 5 copies of $H_s - H$ scaled by a factor s, say $A_1, \dots, A_5$. Each of these copies has a face f_i connecting it to $Q_{1,1}$. Let H_{A_i} be the half-space such that $\partial H_{A_i} \supset f_i$ and $H_{A_i} \supset Q_{1,1}$. Put $F_i = A_i \cup H_{A_i} \cup f_i$. Then F_i is a copy of H_s scaled by a factor s. Thus, by scaling, we have

$$\int_{A_i} dx\, u_{F_i}(x;t) = s^3\int_{H_s - H} dx\, u_{H_s}(x;t/s^2) = s^3E(t/s^2).$$

Define $G_i = \{x \in H_s : d(x, A_i) < \tilde{\epsilon}\}$, where $\tilde{\epsilon} = \frac{s(1-2s-s^2)}{2(1-s)}$. Applying Lemma 5 with $\tilde{D} = H_s$, $F = F_i$, $E = A_i$ and $G = G_i$, we have

$$\int_{A_i} dx\, u_{H_s}(x;t) = \int_{A_i} dx\, u_{F_i}(x;t) + O(e^{-\tilde{\epsilon}^2/(8t)}),\ t \downarrow 0.$$

Hence

$$\begin{aligned}\int_{H_s-(H\cup Q_{1,1})} dx\, u_{H_s}(x;t) &= 5\int_{A_i} dx\, u_{H_s}(x;t)\\ &= 5\int_{A_i} dx\, u_{F_i}(x;t) + O(e^{-\tilde{\epsilon}^2/(8t)})\\ &= 5s^3 E(t/s^2) + O(e^{-\tilde{\epsilon}^2/(8t)}),\ t\downarrow 0.\end{aligned}$$

Combining this with Lemma 13 gives the result. □

We must now consider the different regimes for s.

Lemma 15. *Let $d = \frac{3}{2} + \frac{1}{2}\frac{\log 5}{\log s}$ and fix $0 < s < \sqrt{2}-1, s \neq \frac{1}{5}$. Then there exists a periodic, continuous function $p_s : \mathbb{R} \to \mathbb{R}$ with period $\log(s^{-2})$ such that as $t \downarrow 0$,*

$$E(t) = \frac{s^3}{1-5s^3} - \frac{5s^2(1-s^2)}{1-5s^2}\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi(1-5s)}t + p_s(\log t)t^d + O(t^{3/2}(\log(t^{-1}))^{3/2}). \tag{59}$$

Proof. Define

$$q_s(t)t^d = E(t) - \frac{s^3}{1-5s^3} + \frac{5s^2(1-s^2)}{1-5s^2}\frac{t^{1/2}}{\sqrt{\pi}} - \frac{12s}{\pi(1-5s)}t. \tag{60}$$

Substitute (58) into (60) to obtain

$$q_s(t)t^d = 5s^3 E\left(\frac{t}{s^2}\right) - 5s^3\left(\frac{s^3}{1-5s^3}\right) + 25s^4\left(\frac{1-s^2}{1-5s^2}\right)\frac{t^{1/2}}{\sqrt{\pi}} - 5s\left(\frac{12s}{\pi(1-5s)}\right)t + h(t).$$

From (60), we also have

$$q_s\left(\frac{t}{s^2}\right)\frac{t^d}{5s^3} = E\left(\frac{t}{s^2}\right) - \frac{s^3}{1-5s^3} + \frac{5s(1-s^2)}{1-5s^2}\frac{t^{1/2}}{\sqrt{\pi}} - \frac{12}{\pi s(1-5s)}t,$$

which implies

$$q_s\left(\frac{t}{s^2}\right)t^d = q_s(t)t^d - h(t),$$

or equivalently,

$$q_s(t) = q_s\left(\frac{t}{s^2}\right) + h(t)t^{-d}.$$

Similarly to [6, Proposition 5], define $p_s(\log t)$ by

$$q_s(t) = p_s(\log t) - \sum_{j=1}^{\infty} h(ts^{2j})(ts^{2j})^{-d}. \tag{61}$$

Then

$$q_s\left(\frac{t}{s^2}\right) = p_s\left(\log\frac{t}{s^2}\right) - h(t)t^{-d} - \sum_{j=1}^{\infty} h(ts^{2j})(ts^{2j})^{-d},$$

hence

$$p_s\left(\log\frac{t}{s^2}\right) = q_s(t) + \sum_{j=1}^{\infty} h(ts^{2j})(ts^{2j})^{-d} = p_s(\log t).$$

Since there is a constant $\tilde{C} > 0$ such that $|h(t)| \le \tilde{C}t^{3/2}(\log(t^{-1}))^{3/2}$ for $t \le s^2$, there is a constant $\hat{C} > 0$ such that

$$\begin{aligned}\left|\sum_{j=1}^{\infty} h(ts^{2j})(ts^{2j})^{-d}\right| &\le \hat{C}t^{(3/2)-d}\left((\log(t^{-1}))^{3/2}\sum_{j=1}^{\infty}(s^{3-2d})^j + (-\log s)^{3/2}\sum_{j=1}^{\infty}(2j)^{3/2}(s^{3-2d})^j\right)\\ &= O(t^{(3/2)-d}(\log(t^{-1}))^{3/2}).\end{aligned} \tag{62}$$

Combining (60) with (61) and (62) gives (59). □

Lemma 16. *Let $s = \frac{1}{5}$. Then there exists a periodic, continuous function $p_{\frac{1}{5}} : \mathbb{R} \to \mathbb{R}$ with period $\log 25$ such that as $t \downarrow 0$,*

$$E(t) = \frac{1}{120} - \frac{6}{25}\frac{t^{1/2}}{\sqrt{\pi}} - \frac{6}{5\pi\log 5}t\log t + \frac{12}{5\pi}t + p_{\frac{1}{5}}(\log t)t + O(t^{3/2}(\log(t^{-1}))^{3/2}).$$

Proof. For $s = \frac{1}{5}$, $d = 1$. Define

$$q_{\frac{1}{5}}(t)t = E(t) - \frac{1}{120} + \frac{6}{25}\frac{t^{1/2}}{\sqrt{\pi}} + \frac{6}{5\pi \log 5} t \log t - \frac{12}{5\pi} t. \tag{63}$$

Substitute (58) with $s = \frac{1}{5}$ into (63) to obtain

$$q_{\frac{1}{5}}(t)t = \frac{1}{25}E(25t) - \frac{1}{3000} + \frac{6}{125}\frac{t^{1/2}}{\sqrt{\pi}} + \frac{6}{5\pi \log 5} t \log t + h(t).$$

By considering (63) with t replaced by $25t$, we obtain that

$$q_{\frac{1}{5}}(25t)t = q_{\frac{1}{5}}(t)t - h(t),$$

or equivalently,

$$q_{\frac{1}{5}}(t) = q_{\frac{1}{5}}(25t) + h(t)t^{-1}. \tag{64}$$

Define $p_{\frac{1}{5}}(\log t)$ by

$$q_{\frac{1}{5}}(t) = p_{\frac{1}{5}}(\log t) - \sum_{j=1}^{\infty} h(t(25)^{-j})(t(25)^{-j})^{-1}. \tag{65}$$

Then, as in the proof of Lemma 15, we have $p_{\frac{1}{5}}(\log t) = p_{\frac{1}{5}}(\log 25t)$ via (64). Since (62) holds for $s = \frac{1}{5}$, we obtain the same remainder estimate as in Lemma 15. □

Combining Lemma 12, (57) and Lemma 15, Lemma 16 we obtain (16), (17) respectively. Using the fact that $t \mapsto H_{D_s}(t)$ is continuous, it can be shown that $t \mapsto p_s(\log t)$ is continuous for $0 < s < \sqrt{2} - 1$, see [6, Section 5.1].

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